LUND UNIVERSITY

Department of
AUTOMATIC CONTROL

FRTN10 Multivariable Control

Exam 2016-10-25, 08:00–13:00

Points and grades

All answers must include a clear motivation and a well-formulated answer. Answers may be given in English or Swedish. The total number of points is 25. The maximum number of points is specified for each subproblem.

Accepted aid

The textbook *Glad & Ljung*, standard mathematical tables like TEFYMA, an authorized "Formelsamling i Reglerteknik"/"Collection of Formulas" and a pocket calculator. Hand-outs of lecture notes and lecture slides (including markings/notes) are also allowed.

Results

The result of the exam will be entered into LADOK. The result as well as solutions will be available on the course home page:
`http://www.control.lth.se/course/FRTN10`

1. Let

$$G(s) = \begin{pmatrix} \dfrac{s}{s+1} & \dfrac{-2(s+1)}{s+2} \\ 3 & \dfrac{1}{s+2} \end{pmatrix}$$

a. Compute the poles of $G(s)$. Also state their multiplicity. (1 p)

b. Compute the (transmission) zeros of $G(s)$. Do they impose any fundamental performance limitations? (1 p)

2. Consider the stable system

$$\dot{x} = \begin{pmatrix} -1 & 0 \\ -0.1 & -5 \end{pmatrix} x + \begin{pmatrix} 6 & 2 \\ 0 & 1 \end{pmatrix} u$$
$$y = Cx$$

a. Calculate the controllability Gramian S_x of the system. Is the system controllable? Which state is more difficult to control? (2 p)

b. The observability Gramian of the system is

$$O_x = \begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix}$$

Calculate the Hankel singular values of the system. Which state would you truncate and how large could the error

$$\frac{||y - y_r||_2}{||u||_2}$$

become if we were to keep the static properties of the system, but reduce it to first order? Here, y_r denotes the output signal from the reduced order system. (2 p)

3. A commonly used multivariable model of a distillation column was derived by Wood and Berry in 1973. The model is

$$G(s) = \begin{pmatrix} \dfrac{12.8e^{-s}}{1+16.7s} & \dfrac{-18.9e^{-3s}}{1+21s} \\ \dfrac{6.6e^{-7s}}{1+10.9s} & \dfrac{-19.4e^{-3s}}{1+14.4s} \end{pmatrix}$$

a. Calculate the Relative Gain Array in stationarity and decide how the input–output pairing should be done for decentralized control. Will there be visible interaction between the two control loops? (2 p)

b. Find suitable matrices W_1 and W_2 such that $\tilde{G} = W_2 G W_1$ is decoupled in stationarity. (1 p)

c. For each of the four statements below, explain whether it is true or false. (2 p)

i. After the decoupling in subproblem **b**, there will be no visible cross-coupling in the closed-loop step responses.

ii. By finding suitable decoupling matrices W_1 and W_2 we modify our physical process so that it becomes diagonal.

iii. The fundamental limitations imposed by the process deadtimes can be removed by properly chosen decoupling matrices.

iv. The decoupling approach allows us to design SISO controllers while still taking the MIMO process information into account.

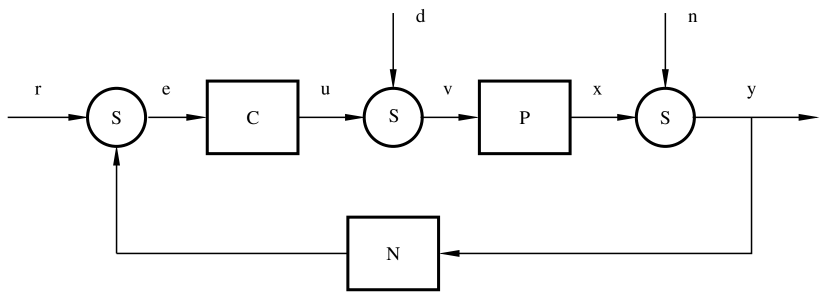

Figure 1 MIMO control system in Problem 4.

4. Consider the MIMO control system in Figure 1, where P and C are the (matrix) transfer functions of the process and the controller, respectively.

a. Calculate the (matrix) transfer function from $\begin{pmatrix} r \\ d \end{pmatrix}$ to u in terms of P and C. (1 p)

b. Suppose that $P(s)$ has 2 inputs and 3 outputs. What dimensions must then r, d and $C(s)$ have? (1 p)

c. The singular value (sigma) plots of P and C are shown in Figure 2. Can stability of the closed-loop system be guaranteed using the Small Gain Theorem? (1 p)

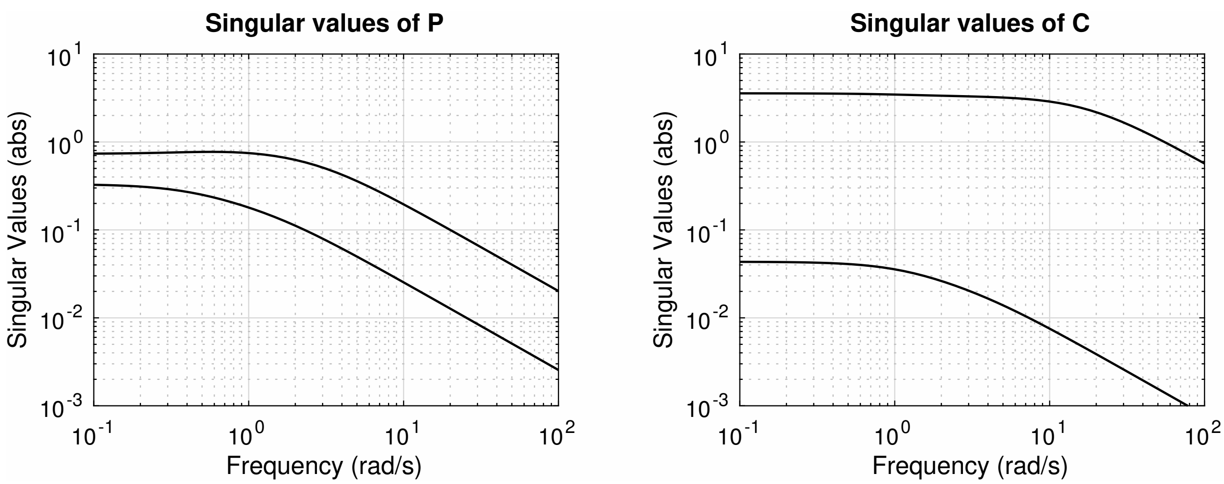

Figure 2 Sigma plots in Problem 4c.

5. A young student who has only taken a basic course in control has attempted to design a controller for the process

$$P(s) = \frac{2-s}{s(s^2+5s+12)}$$

The controller $C(s)$ was designed as a state feedback from an observer. The control poles were placed in $-7 \pm i$ and -8 and the observer poles in $-14 \pm 2i$ and -16.

The student proudly proclaims: "Look at these poles! I have designed a very fast and well-damped closed-loop system!".

You become suspicious and ask the student to plot the magnitude of the sensitivity function S. The result in shown in Figure 3.

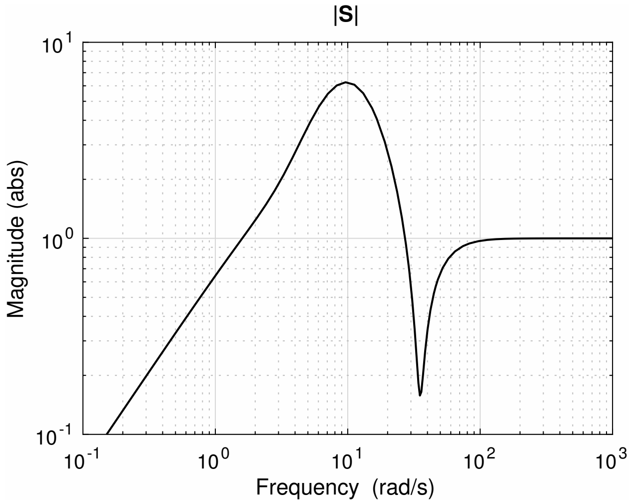

Figure 3 Magnitude plot of the sensitivity function in Problem 5.

a. The student has forgotten all about the sensitivity function and its interpretation. Explain to him/her why this sensitivity function is a sign of very poor robustness. Also explain how you could immediately realize that it should not be possible to design a very fast closed-loop system for this plant. (2 p)

b. Using the sensitivity weighting function

$$W_s(s) = \frac{s + M_s\omega_0}{M_s s}, \quad M_s,\, \omega_0 > 0$$

show that the specification

$$|S(i\omega)| \leq |W_s^{-1}(i\omega)|, \quad \forall\omega$$

is impossible to fulfill for any value of M_s if $\omega_0 > 2$. For what values of ω_0 is it impossible to fulfill $M_s \leq 1.4$? (2 p)

6. A controller derived from the standard LQG framework will not automatically feature integral action. One way of approximating it is to add a noise model to the Kalman filter where the noise is assumed to have a very high spectral density for low frequencies. Assuming that the initial model of the system is

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + v_1(t) \\ z(t) &= Cx(t) \\ y(t) &= Cx(t) + v_2(t) \end{aligned}$$

we would like to extend it to

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + v_1(t) \\ z(t) &= Cx(t) + w(t) \\ y(t) &= Cx(t) + v_2(t) + w(t) \end{aligned}$$

where w is noise with high spectral density for low frequencies. We model w as white noise n filtered through $H(s) = \frac{1}{s+\delta}$, i.e.

$$w = Hn$$

Here, $\delta > 0$ is some small number.

a. Explain why we do not use a pure integrator, i.e. $H(s) = \frac{1}{s}$, in our noise model when designing an LQG controller. (1 p)

b. Extend the process model with the noise model, such that it attains the form

$$\begin{aligned}\dot{x}_e(t) &= A_e x_e(t) + B_e u(t) + v_{1e}(t)\\ z(t) &= C_e x_e(t)\\ y(t) &= C_e x_e(t) + v_2(t)\end{aligned}$$

where $v_{1e} = [\,v_1^T \quad n\,]^T$. Express A_e, B_e and C_e in A, B and C. (1 p)

c. Now assume that $A = -1$, $B = 1$, $C = 1$ and $\delta = 0.001$. Which one of the three controllers A, B or C in Figure 4 could be the transfer function of an LQG controller based on the extended model? You can assume that the low- and high-frequency asymptotes of the controllers are visible in the plot. (1 p)

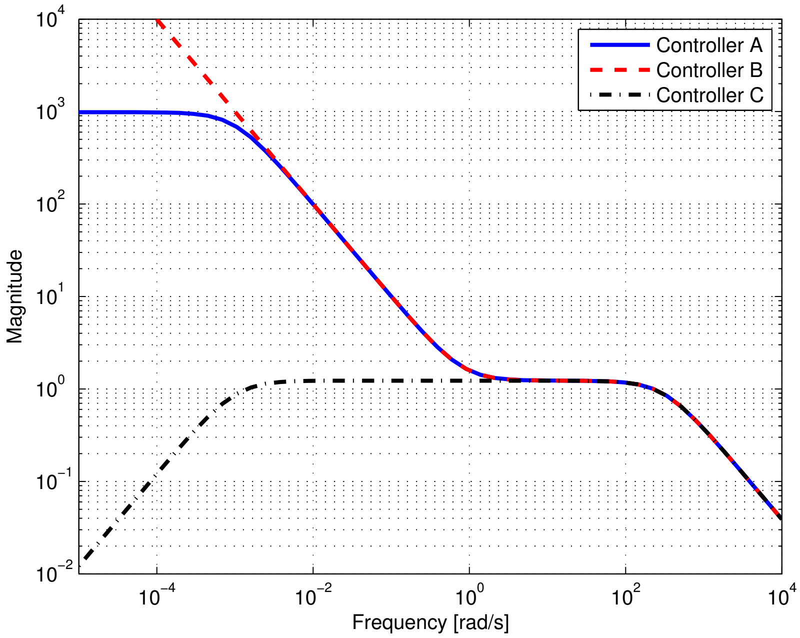

Figure 4 Bode diagram of three possible candidates for the LQG controller in Problem 6.

7. Consider the closed-loop system in Figure 5, where P_0 is a SISO system. A controller should be designed to attenuate the effect of input disturbances d on the process output x, while keeping the effect of measurement noise n on the control signal u to a minimum.

a. Define $z = [\,x \quad u\,]^T$ and $w = [\,d \quad n\,]^T$, and rewrite the system in Figure 5 to the more general form in Figure 6. Provide the expressions for P_{zw}, P_{zu}, P_{yw} and P_{yu}. (1 p)

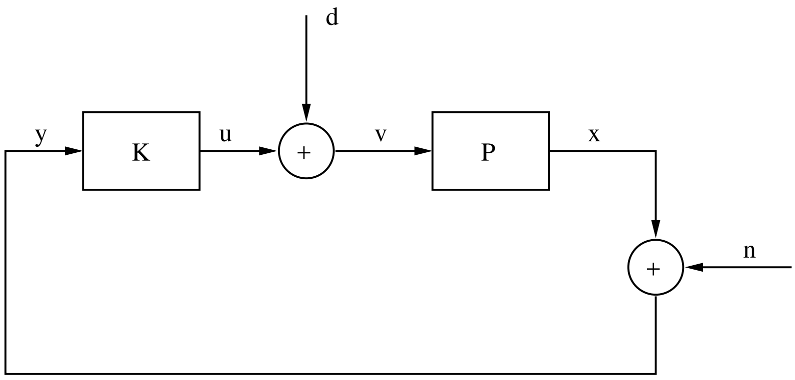

Figure 5 Block diagram of the closed-loop system in Problem 7.

b. Determine the closed-loop system from w to z in terms of P_0 and C. (1 p)

c. Now assume $P_0(s) = \frac{1}{s+2}$. Two controllers, $C_1(s) = \frac{1}{s(s+3)}$ and $C_2(s) = 2$, have been found to achieve stable closed-loop systems. Determine the Q-parameterization for each controller, where the parameter $Q(s)$ should be a stable transfer function. The resulting closed-loop system should be linear in $Q(s)$. (1 p)

d. The response of the closed-loop system is tested for a step input in d and an impulse in n for each controller. Also, the $\mathcal{L}_2$-gain of the closed-loop system is computed for each case. The results are:

Evaluation	C_1	C_2
Maximum value of $x(t)$ after unit step in d.	0.44	0.25
Minimum value of $u(t)$ after unit impulse in n.	-0.14	-2.0
$\mathcal{L}_2$-gain of closed-loop transfer function from n to u.	1.0	0.5

Determine a controller which achieves $x(t) \leq 0.4$ after a unit step in d and $u(t) \geq -0.8$ after an impulse in n. Additionally, the maximum allowable $\mathcal{L}_2$-gain of the transfer function from n to u is 0.95.

Note: You can express your controller in terms of the Q-parameters $Q_1(s)$ and $Q_2(s)$ for the controllers $C_1(s)$ and $C_2(s)$ respectively. (1 p)

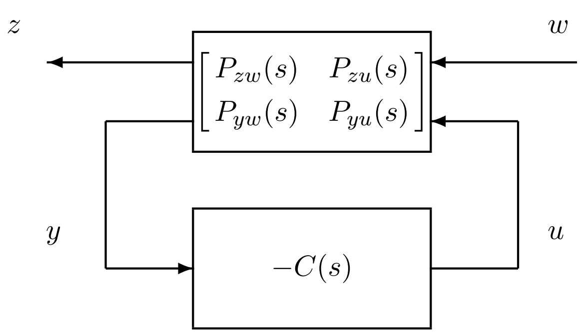

Figure 6 General closed-loop system in Problem 7.