



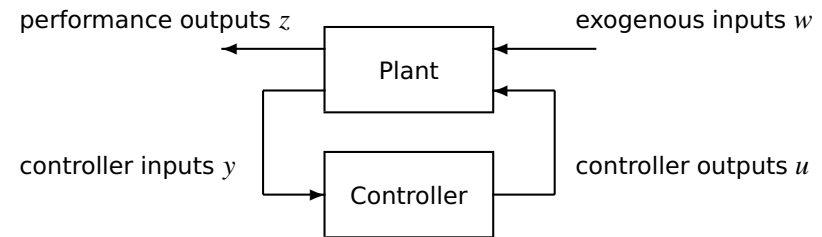
Lecture 9 – Outline

Linear-quadratic control:

- 1 Dynamic programming
- 2 The Riccati equation
- 3 Optimal state feedback
- 4 Stability and robustness



A general optimization setup



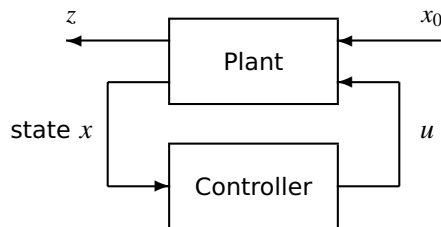
General objective: find a controller that optimizes the closed-loop system $G_{zw}(s)$.

Lectures 9–11: Problems with analytic solutions

Lectures 12–14: Problems with numeric solutions



Today's problem: Optimal state feedback



Optimization problem:

$$\text{minimize } J = \int_0^\infty |z|^2 dt = \int_0^\infty \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^T \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} dt$$

$$\text{subject to } \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

$$Q = \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix} > 0 \text{ is a symmetric matrix (design parameter)}$$



Why linear-quadratic control?

- Simple, analytic solution
 - Quadratic cost functions give linear state feedback control laws
- Always stabilizing
- Works for MIMO systems
- Guaranteed robustness (in the state feedback case)
- Foundation for more advanced methods like model-predictive control (MPC)



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Dynamic programming: example

Determine u_0 and u_1 if the objective is to minimize

$$x_1^2 + x_2^2 + u_0^2 + u_1^2$$

when

$$x_1 = x_0 + u_0$$

$$x_2 = x_1 + u_1$$

Hint: Go backwards in time.



Dynamic programming: example

Break the problem into smaller parts that can be solved sequentially:

$$\min_{u_0, u_1} \{x_1^2 + x_2^2 + u_0^2 + u_1^2\} = \min_{u_0} \left\{ x_1^2 + u_0^2 + \underbrace{\min_{u_1} \{x_2^2 + u_1^2\}}_{J_1(x_1)}(x_1) \right\}$$

$$J_1(x_1) = \min_{u_1} \{(x_1 + u_1)^2 + u_1^2\} = \min_{u_1} \left\{ 2\left(u_1 + \frac{1}{2}x_1\right)^2 + \frac{1}{2}x_1^2 \right\}$$

$$= \frac{1}{2}x_1^2 \quad \text{with minimum attained for } u_1 = -\frac{1}{2}x_1$$

$$J_0(x_0) = \min_{u_0} \{(x_0 + u_0)^2 + u_0^2 + J_1(x_0 + u_0)\} = \min_{u_0} \left\{ \frac{5}{2}(u_0 + \frac{3}{5}x_0)^2 + \frac{3}{5}x_0^2 \right\}$$

$$= \frac{3}{5}x_0^2 \quad \text{with minimum attained for } u_0 = -\frac{3}{5}x_0$$



Quadratic optimal cost

It can be shown that the optimal cost on a time interval $[t, \infty)$ is quadratic:

$$\min_{u[t, \infty)} \int_t^\infty \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^T Q \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} d\tau = x^T(t) S x(t), \quad S = S^T > 0$$

when

$$\dot{x}(t) = Ax(t) + Bu(t)$$

and

$$Q = \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix} > 0$$



Dynamic programming, Richard E. Bellman, 1957



Bellman's principle of optimality:

An optimal trajectory on the time interval $[t, T]$ must be optimal also on each of the subintervals $[t, t + \epsilon]$ and $[t + \epsilon, T]$.



Bellman's equation

From the previous slide:

$$x^T(t)Sx(t) = \min_{u(t)} \left\{ \left(x^T(t)Q_1x(t) + 2x^T(t)Q_{12}u(t) + u^T(t)Q_2u(t) \right) \epsilon + \left[x(t) + (Ax(t) + Bu(t))\epsilon \right]^T S \left[x(t) + (Ax(t) + Bu(t))\epsilon \right] \right\}$$

Neglecting the ϵ^2 terms gives **Bellman's equation**:

$$0 = \min_{u(t)} \left\{ x^T(t)Q_1x(t) + 2x^T(t)Q_{12}u(t) + u^T(t)Q_2u(t) + 2x^T(t)S(Ax(t) + Bu(t)) \right\}$$



Dynamic programming for linear-quadratic control

For an infinitesimal time step of length ϵ ,

$$x(t + \epsilon) = x(t) + (Ax(t) + Bu(t))\epsilon \quad \text{as } \epsilon \rightarrow 0$$

Invoking the principle of optimality for $[t, t + \epsilon]$ and $[t + \epsilon, \infty]$:

$$\begin{aligned} x^T(t)Sx(t) &= \min_{u[t, \infty)} \int_t^\infty \begin{bmatrix} x(\tau) \\ u(\tau) \end{bmatrix}^T Q \begin{bmatrix} x(\tau) \\ u(\tau) \end{bmatrix} d\tau \\ &= \min_{u[t, \infty)} \left\{ \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T Q \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \epsilon + \int_{t+\epsilon}^\infty \begin{bmatrix} x(\tau) \\ u(\tau) \end{bmatrix}^T Q \begin{bmatrix} x(\tau) \\ u(\tau) \end{bmatrix} d\tau \right\} \\ &= \min_{u(t)} \left\{ x^T(t)Q_1x(t) + 2x^T(t)Q_{12}u(t) + u^T(t)Q_2u(t) \epsilon + \left[x(t) + (Ax(t) + Bu(t))\epsilon \right]^T S \left[x(t) + (Ax(t) + Bu(t))\epsilon \right] \right\} \end{aligned}$$



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Completion of squares

Suppose $Q_u > 0$. Then the quadratic form

$$\begin{aligned} & x^T Q_x x + 2x^T Q_{xu} u + u^T Q_u u \\ &= (u + Q_u^{-1} Q_{xu}^T x)^T Q_u (u + Q_u^{-1} Q_{xu}^T x) + x^T (Q_x - Q_{xu} Q_u^{-1} Q_{xu}^T) x \end{aligned}$$

is minimized by

$$u = -Q_u^{-1} Q_{xu}^T x$$

The minimum is

$$x^T (Q_x - Q_{xu} Q_u^{-1} Q_{xu}^T) x$$



Jocopo Francesco Riccati, 1676–1754



The Riccati equation

Completion of squares in Bellman's equation gives

$$\begin{aligned} 0 &= \min_{u_t} \left\{ x_t^T Q_1 x_t + 2x_t^T Q_{12} u_t + u_t^T Q_2 u_t \right\} + 2x_t^T S (Ax_t + Bu_t) \\ &= \min_{u_t} \left\{ x_t^T [Q_1 + A^T S + SA] x_t + 2x_t^T [Q_{12} + SB] u_t + u_t^T Q_2 u_t \right\} \\ &= x_t^T \left(Q_1 + A^T S + SA - (SB + Q_{12}) Q_2^{-1} (SB + Q_{12})^T \right) x_t \end{aligned}$$

with minimum attained for

$$u_t = -Q_2^{-1} (SB + Q_{12})^T x_t$$

The equation

$$0 = Q_1 + A^T S + SA - (SB + Q_{12}) Q_2^{-1} (SB + Q_{12})^T$$

is called the *algebraic Riccati equation*



Algebraic Riccati equations in Matlab

`care` Solve continuous-time algebraic Riccati equations.

`[X,L,G] = care(A,B,Q,R,S,E)` computes the unique stabilizing solution X of the continuous-time algebraic Riccati equation

$$A^T X E + E^T X A - (E^T X B + S) R^{-1} (B^T X E + S') + Q = 0.$$

When omitted, R , S and E are set to the default values $R=I$, $S=0$, and $E=I$. Beside the solution X , `care` also returns the gain matrix

$$G = R^{-1} (B^T X E + S')$$

and the vector L of closed-loop eigenvalues (i.e., `EIG(A-B*G,E)`).



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Remarks

Note that the optimal control law does not depend on x_0 .

The optimal feedback gain L is static since we are solving an infinite-horizon problem.

(LQ theory can also be applied to finite-horizon problems and problems with time-varying system matrices. We then obtain a Riccati differential equation for $S(t)$ and a time-varying state feedback, $u(t) = -L(t)x(t)$)



Linear-quadratic optimal control

Control problem:

$$\text{Minimize } J = \int_0^\infty \left(x^T(t) Q_1 x(t) + 2x^T(t) Q_{12} u(t) + u^T(t) Q_2 u(t) \right) dt$$

$$\text{subject to } \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

Solution: Assume (A, B) stabilizable (i.e., any unstable modes are controllable). Then there is a unique $S = S^T > 0$ solving the algebraic Riccati equation

$$0 = Q_1 + A^T S + SA - (SB + Q_{12}) Q_2^{-1} (SB + Q_{12})^T$$

The optimal control law is $u = -Lx$ with $L = Q_2^{-1} (SB + Q_{12})^T$.

The optimal cost is $J^* = x_0^T S x_0$.



Example: Control of an integrator

For $\dot{x}(t) = u(t)$, $x(0) = x_0$,

$$\text{Minimize } J = \int_0^\infty \{x(t)^2 + \rho u(t)^2\} dt$$

$$\text{Riccati equation } 0 = 1 - S^2/\rho \Rightarrow S = \sqrt{\rho}$$

$$\text{Controller } L = S/\rho = 1/\sqrt{\rho} \Rightarrow u = -x/\sqrt{\rho}$$

$$\text{Closed loop system } \dot{x} = -x/\sqrt{\rho} \Rightarrow x = x_0 e^{-t/\sqrt{\rho}}$$

$$\text{Optimal cost } J^* = x_0^T S x_0 = x_0^2 \sqrt{\rho}$$

What values of ρ give the fastest response? Why?



Solving the LQ problem in Matlab

`lqr` Linear-quadratic regulator design for state space systems

`[K,S,E] = lqr(SYS,Q,R,N)` calculates the optimal gain matrix K such that:

* For a continuous-time state-space model SYS , the state-feedback law $u = -Kx$ minimizes the cost function

$$J = \text{Integral} \{x'Qx + u'Ru + 2x'Nu\} dt$$

subject to the system dynamics $dx/dt = Ax + Bu$

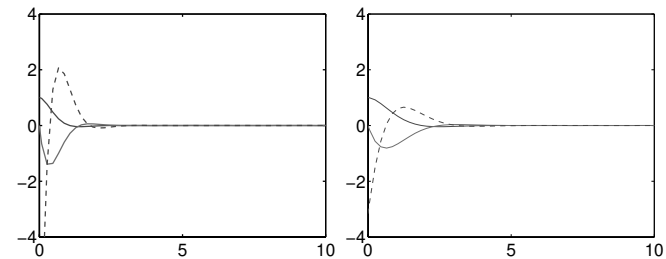
The matrix N is set to zero when omitted. Also returned are the solution S of the associated algebraic Riccati equation and the closed-loop eigenvalues $E = \text{EIG}(A-B*K)$.



Example – Double integrator

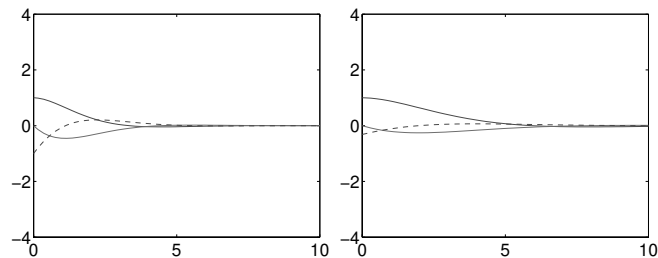
$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad Q_2 = \rho \quad x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

States (full) and input (dotted) for $\rho = 0.01, \rho = 0.1$:

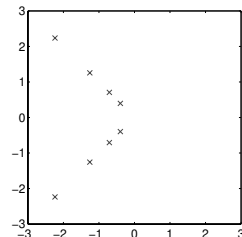


Example – Double integrator

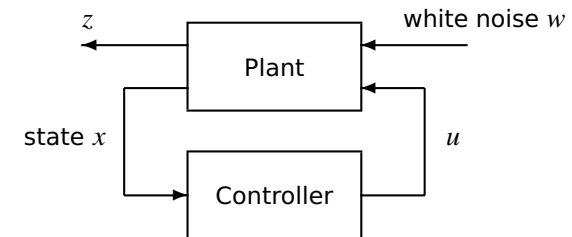
States (full) and inputs (dotted) for $\rho = 1, \rho = 10$:



Closed loop poles:
 $s = 2^{-1/2} \rho^{-1/4} (-1 \pm i)$



Stochastic interpretation of LQ control



$$\begin{aligned} \text{Minimize} \quad & J = E |z|^2 = E \{x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u\} \\ \text{subject to} \quad & \dot{x}(t) = Ax(t) + Bu(t) + w(t) \end{aligned}$$

where w is white noise with intensity R . Same Riccati equation and solution (S, L) as in the deterministic case. The optimal cost is

$$J^* = E x^T S x = \text{trace}(SR)$$



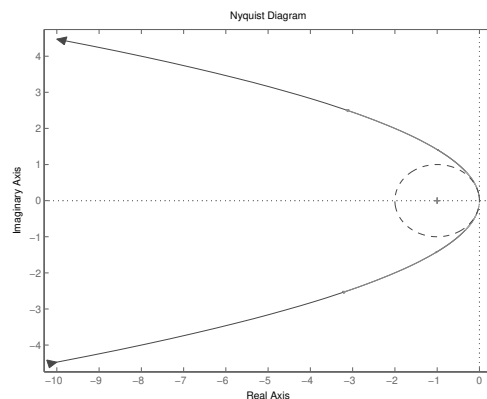
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Robustness of optimal state feedback



The distance from the loop gain $L(i\omega I - A)^{-1}B$ to -1 is never smaller than 1. This is always true(!) when $Q_1 > 0$, $Q_{12} = 0$ and $Q_2 > 0$ is scalar. The phase margin is at least 60° and the gain margin is infinite!



Stability of the closed-loop system

Assume that

$$Q = \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix} > 0$$

and that there exists a solution $S > 0$ to the algebraic Riccati equation. Then the optimal controller $u(t) = -Lx(t)$ gives an asymptotically stable closed-loop system $\dot{x}(t) = (A - BL)x(t)$.

Proof:

$$\begin{aligned} \frac{d}{dt}x^T(t)Sx(t) &= 2x^T S \dot{x} = 2x^T S(Ax + Bu) \\ &= -\left(x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u\right) < 0 \text{ for } x(t) \neq 0 \end{aligned}$$

Hence $x^T(t)Sx(t)$ is decreasing and tends to zero as $t \rightarrow \infty$.



Lecture 9 – summary

- We specify what “optimal control” means using a quadratic cost function.
- Solving an algebraic Riccati equation gives the optimal state feedback law $u = -Lx$:

$$\begin{aligned} 0 &= Q_1 + A^T S + SA - (SB + Q_{12})Q_2^{-1}(SB + Q_{12})^T \Rightarrow S \\ L &= Q_2^{-1}(SB + Q_{12})^{-1} \end{aligned}$$

- The LQ controller has remarkable robustness properties.