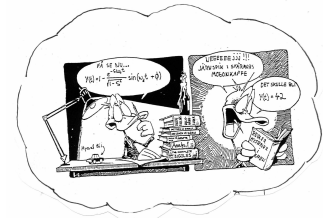


- L1-L5 Specifications, models and loop-shaping by hand
 1. Introduction and system representations
 2. Stability and robustness
 3. Specifications and disturbance models
 4. Control synthesis in frequency domain
 5. Case study
- L6-L8 Limitations on achievable performance
- L9-L11 Controller optimization: Analytic approach
- L12-L14 Controller optimization: Numerical approach

- Introduction/examples
- Overview of course
- Review linear systems
 - Review of time-domain models
 - Review of frequency-domain models
 - Norm of signals
 - Gain of systems



Lecture 2: Stability and Robustness

- Stability
- Robustness and sensitivity
- Small gain theorem

Demo: "Inverted pendulum"

Stability is crucial

- bicycle
- JAS 39 Gripen
- Mercedes A-class
- ABS brakes

Stability of autonomous systems

The autonomous system

$$\frac{dx}{dt} = Ax(t)$$

is called exponentially stable if the following equivalent conditions hold

1. There exist constants $\alpha, \beta > 0$ such that

$$|x(t)| \leq \alpha e^{-\beta t} |x(0)| \quad \text{for } t \geq 0$$

2. All eigenvalues of A are in the left half plane (LHP), that is all eigenvalues have negative real part.
3. All roots of the polynomial $\det(sI - A)$ are in the LHP.

Stability of input-output maps

The transfer function $G(s)$ of a continuous time system, is said to be input-output stable (I/O-stable, or often just called "stable") if the following equivalent conditions hold:

- All poles of G have negative real part (G is Hurwitz stable)
- The impulse response of G decays exponentially.

Warning: There may be unstable pole-zero cancellations (which also render the system either uncontrollable and/or unobservable) and these may not be seen in the transfer function!!

For discrete time systems the corresponding conditions are : a pulse transfer function $G(z)$ of a discrete time system

- All poles of G are inside the unit circle (G is Schur stable).
- The pulse response of G decays exponentially.

Eigenvalues determine stability

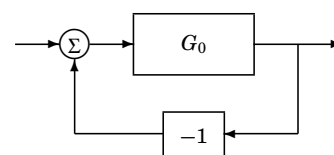
The matrix A can always be written on the form

$$A = U \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} U^{-1}. \quad \text{Hence } e^{At} = U \begin{bmatrix} e^{\lambda_1 t} & & * \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} U^{-1}.$$

The number $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A .

e^{At} decays exponentially if and only if $\operatorname{Re}\{\lambda_k\} < 0$ for all k .

Stability of feedback loops



The closed loop system is input-output stable if and only if all solutions to the equation

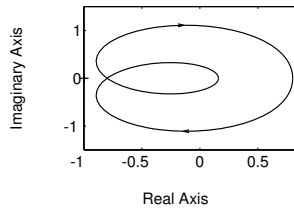
$$1 + G_0(s) = 0$$

are in the left half plane (i.e. has negative real part).

The Nyquist criterion

If $G_0(s)$ is stable, then the closed loop system $[1 + G_0(s)]^{-1}$ is stable if and only if the Nyquist curve does not encircle -1

The difference between the number of unstable poles in $[1 + G_0(s)]^{-1}$ and the number of unstable poles in $G_0(s)$ is equal to the number of times the point -1 is encircled by the Nyquist plot in the clockwise direction.



NOTE: nyquist-plot cmd in Matlab plots for both positive and negative frequencies!

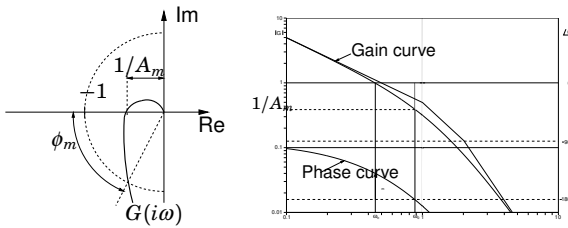
Amplitude and phase margin

Amplitude margin A_m

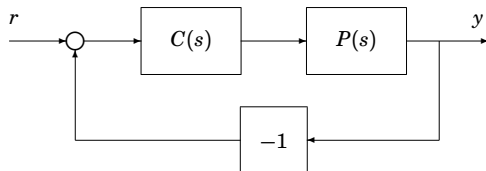
$$\arg G(i\omega_0) = -180^\circ, \quad |G(i\omega_0)| = \frac{1}{A_m}$$

Phase margin ϕ_m

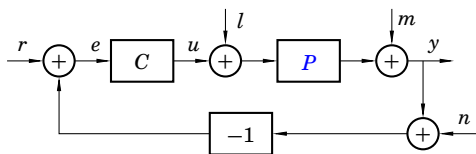
$$|G(i\omega_c)| = 1, \quad \arg G(i\omega_c) = \phi_m - 180^\circ$$



How sensitive is T to changes in P ?



$$Y(s) = \underbrace{\frac{P(s)C(s)}{1 + P(s)C(s)}}_{T(s)} R(s)$$



Note that the

- complementary sensitivity function T is the transfer function $G_{r \rightarrow y}$
- sensitivity function S is the transfer function $G_{m \rightarrow y}$

$$S + T = 1$$

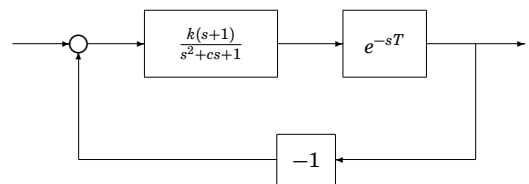
Note: there are **four different transfer functions** for this closed-loop system and all have to be stable for the system to be stable!

It may be OK to use an unstable controller C

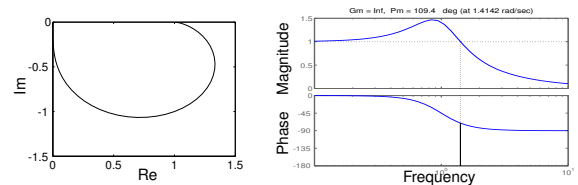
Sensitivity and Robustness

- How sensitive is the closed loop system to model errors?
- How do we measure the "distance to instability"?
- Is it possible to guarantee stability for all systems within some distance from the ideal model?

Mini-problem



Nominally $k = 1$, $c = 1$ and $T = 0$. How much margin is there in each of the parameters before the system becomes unstable?



$$\frac{dT}{dP} = \frac{d}{dP} \left(1 - \frac{1}{1 + PC} \right) = \frac{C}{(1 + PC)^2} = \frac{T}{P(1 + PC)}$$

Define the sensitivity function, S :

$$S := \frac{d(\log T)}{d(\log P)} = \frac{dT/T}{dP/P} = \frac{1}{1 + PC}$$

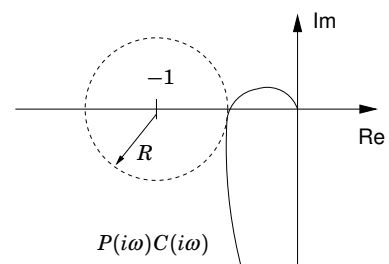
and the complementary sensitivity function T :

$$T := 1 - S = \frac{PC}{1 + PC}$$

Nyquist plot illustration

The sensitivity function measures the distance from the Nyquist plot to -1 .

$$R^{-1} = \sup_{\omega} \left| \frac{1}{1 + P(i\omega)C(i\omega)} \right|$$



- Stability
- Robustness and sensitivity
- Small gain theorem

Definition of matrix norm

For $M \in \mathbf{R}^{n \times n}$, we use the " L_2 -induced norm"

$$\|M\| := \sup_x \frac{|Mx|}{|x|} = \sup_x \sqrt{\frac{x^T M^T M x}{x^T x}} = \sqrt{\bar{\lambda}(M^T M)}$$

Here $\bar{\lambda}(M^T M)$ denotes the largest eigenvalue of $M^T M$. The fraction $|Mx|/|x|$ is maximized when x is a corresponding eigenvector.

Example

Matlab-code for singular value decomposition of the matrix

$$A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$$

SVD :

$$A = U \cdot S \cdot V^*$$

where both the matrices U and V are unitary (i.e. have orthonormal columns s.t. $V^* \cdot V = I$) and S is the diagonal matrix with (sorted decreasing) singular values σ_i .

Multiplying A with an input vector along the first column in V gives

$$A \cdot V_{(:,1)} = USV^* \cdot V_{(:,1)} = US \begin{bmatrix} 1 \\ 0 \end{bmatrix} = U_{(:,1)} \cdot \sigma_1$$

That is, we get maximal gain σ_1 in the output direction $U_{(:,1)}$ if we use an input in direction $V_{(:,1)}$ (and minimal gain $\sigma_n = \sigma_2$ if we use the last column $V_{(:,n)} = V_{(:,2)}$).

```
>> A=[2 4 ; 0 3]
A =
     2     4
     0     3
>> [U,S,V]=svd(A)
U =
    0.8416   -0.5401
    0.5401    0.8416
S =
    5.2631         0
         0    1.1400
V =
    0.3198   -0.9475
    0.9475    0.3198
>> A*V(:,1)
ans =
    4.4296
    2.8424
>> U(:,1)*S(1,1)
ans =
    4.4296
    2.8424
```

Miniproblem

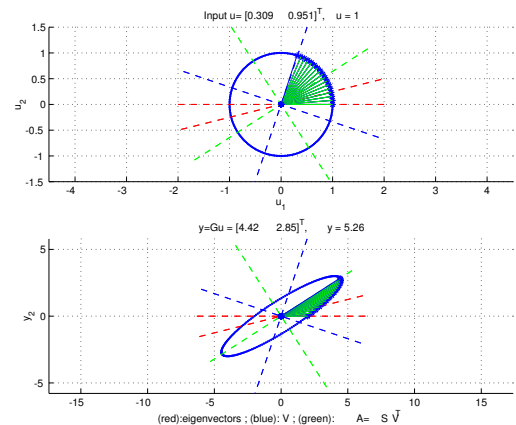
What are the gains of the following systems?

1. $y(t) = -u(t)$ (a sign shift)
2. $y(t) = u(t - T)$ (a time delay)
3. $y(t) = \int_0^t u(\tau) d\tau$ (an integrator)
4. $y(t) = \int_0^t e^{-(t-\tau)} u(\tau) d\tau$ (a first order filter)

For $x \in \mathbf{R}^n$, we use the " L_2 -norm"

$$|x| = \sqrt{x^T x} = \sqrt{x_1^2 + \dots + x_n^2}$$

Different gains in different directions: $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$



Example: matlab-demo

The L_2 -norm of a signal

For $y(t) \in \mathbf{R}^n$ the " L_2 -norm"

$$\|y\|_2 := \sqrt{\int_0^\infty |y(t)|^2 dt} \quad \text{is equal to} \quad \sqrt{\frac{1}{2\pi} \int_{-\infty}^\infty |\mathcal{L}y(i\omega)|^2 d\omega}$$

The equality is known as Parseval's formula

The L_2 -gain of a system For a system S with input u and output $S(u)$, the L_2 -gain is defined as

$$\|S\| := \sup_u \frac{\|S(u)\|_2}{\|u\|_2}$$

The L_2 -gain from frequency data

Consider a stable system S with input u and output $S(u)$ having the transfer function $G(s)$. Then, the system gain

$$\|S\| := \sup_u \frac{\|S(u)\|_2}{\|u\|_2} \quad \text{is equal to} \quad \|G\|_\infty := \sup_\omega |G(i\omega)|$$

Proof. Let $y = S(u)$. Then

$$\|y\|^2 = \frac{1}{2\pi} \int_{-\infty}^\infty |\mathcal{L}y(i\omega)|^2 d\omega = \frac{1}{2\pi} \int_{-\infty}^\infty |G(i\omega)|^2 \cdot |\mathcal{L}u(i\omega)|^2 d\omega \leq \|G\|_\infty^2 \|u\|^2$$

The inequality is arbitrarily tight when $u(t)$ is a sinusoid near the maximizing frequency.

Example: Consider the transfer function matrix $G(i\omega)$

$$G(s) = \begin{bmatrix} \frac{2}{s+1} & \frac{4}{2s+1} \\ \frac{s}{s^2+0.1s+1} & \frac{3}{s+1} \end{bmatrix}$$

```
>> s=tf('s')
>> G=[ 2/(s+1) 4/(2*s+1); s/(s^2+0.1*s+1) 3/(s+1)];
>> sigma(G) % plot sigma values of G wrt fq
>> grid on
>> norm(G,inf) % infinity norm = system gain
ans =
    10.3577
```

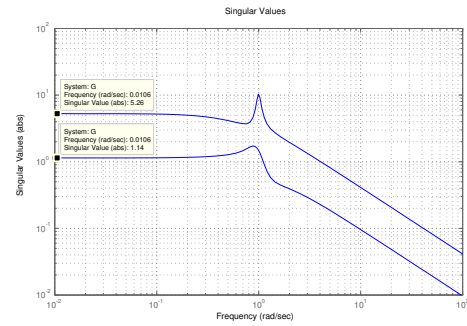
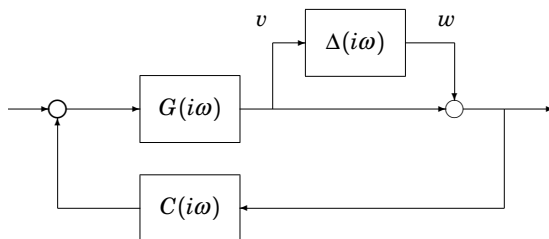


Figure : The singular values of the transfer function matrix (prev slide). Note that $G(0)=[2, 4; 0, 3]$ which corresponds to M in the SVD-example above. $\|G\|_\infty = 10.3577$.

Robustness

How large perturbations $\Delta(i\omega)$ can be tolerated without instability?



Proof

Define $\|y\|_T = \sqrt{\int_0^T |y(t)|^2 dt}$. Then $\|S(y)\|_T \leq \|S\| \cdot \|y\|_T$.

$$\begin{aligned} e_1 &= r_1 + S_2(r_2 + S_1(e_1)) \\ \|e_1\|_T &\leq \|r_1\|_T + \|S_2\| (\|r_2\|_T + \|S_1\| \cdot \|e_1\|_T) \\ \|e_1\|_T &\leq \frac{\|r_1\|_T + \|S_2\| \cdot \|r_2\|_T}{1 - \|S_1\| \cdot \|S_2\|} \end{aligned}$$

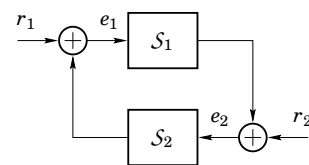
This shows bounded gain from (r_1, r_2) to e_1 .

The gain to e_2 is bounded in the same way.

Lecture 2

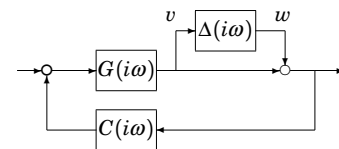
- Stability
- Robustness and sensitivity
- Small gain theorem

The Small Gain Theorem



Assume that S_1 and S_2 are input-output stable. If $\|S_1\| \cdot \|S_2\| < 1$, then the gain from (r_1, r_2) to (e_1, e_2) in the closed loop system is finite.

Application to robustness analysis



The transfer function from w to v is

$$\frac{C(i\omega)G(i\omega)}{1 + C(i\omega)G(i\omega)}$$

Hence the small gain theorem guarantees stability if

$$\|\Delta\|_\infty < \left(\sup_\omega \left\| \frac{C(i\omega)G(i\omega)}{1 + C(i\omega)G(i\omega)} \right\| \right)^{-1}$$