Bicycle Dynamics and Control

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Thanks to Richard Klein and Anders Lennartsson

Introduction

- Bicycles are convenient, environmental friendly, and efficient transportation devices
- Not trivial to explain how bicycles work. Example: Do you actively stabilize a bicycle when you ride it?
- Good illustration of many interesting issues in control
  - Modeling, stabilization, RHP poles & zeros
  - Fundamental limitations
  - Integrated process and control design
- Experiences from using bicycles in education?
  - Motivation
  - Concrete illustration of ideas and concepts
  - Many (simple and advanced) experiments
- Very high student attraction
- Klein's adapted bicycles for children with disabilities

Bicycles in Science

- W. J. Macquorn Rankine 1869 - famous thermodynamicist - counter-steering
- E. Carvallo 1898-1900 Prix Fourneyron
- F. J. W. Whipple 1899
- Felix Klein and Arnold Sommerfeld 1910
- D. E. H. Jones 1942 The stability of the bicycle. Physics Today, reissued 2006
- Ju. I. Neimark and N. A. Fufaev 1972 (1967) Dynamics of nonholonomic systems AMS

Bicycle Dynamics and Control

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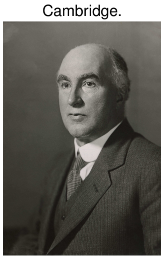
3. Stabilization
4. Rear wheel steering
5. Steering and stabilization
6. More Complex Models
7. Experiments
8. Conclusions

Bicycle Modeling

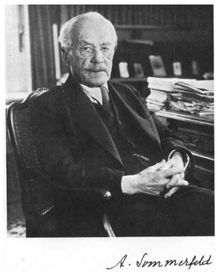
- Geometry, tires, elasticities, rider
- Early models Whipple and Carvallo 1899-1900: 4th order models
- Timoshenko-Young 1920 2nd order
- Popular thesis topics 1960-1980, manual derivations
- Rider models
- Motorcycle models Sharp 1970
- The role of software for symbolic calculation, multi-body programs and Modelica
- The control viewpoint, bicycle robots

Whipple developed his 4th order model as an undergraduate at Cambridge.



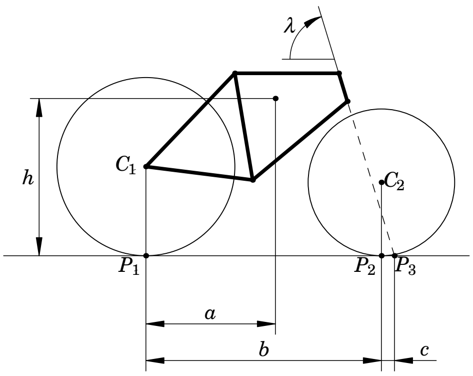
Arnold Sommerfeld on Gyroscopic Effects

That the gyroscopic effects of the wheels are very small can be seen from the construction of the wheel: if one wanted to strengthen the gyroscopic effects, one should provide the wheels with heavy rims and tires instead of making them as light as possible. It can nevertheless be shown that these weak effects contribute their share to the stability of the system.

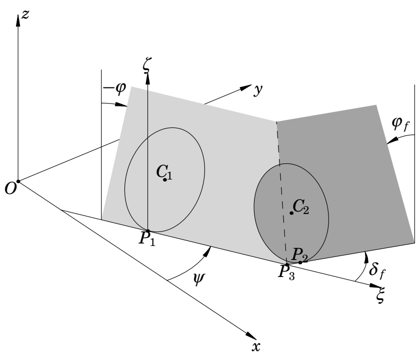


Four of Sommerfeld's graduate students got the Nobel Prize Heisenberg 1932, Debye 1936, Pauli 1945 and Bethe 1962

Geometry



Coordinate System



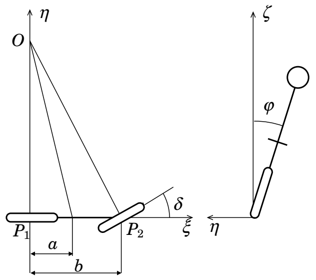
Tilt Dynamics

Assume all angles small. Angular momentum and torque along ζ axis

$$\mathcal{M}_\xi = J_{\xi\zeta}\omega_\xi - D_{\xi\zeta}\omega_\zeta = J\frac{d\phi}{dt} - D\frac{V_0}{b}\delta$$

$$T_\xi = mg\ell\phi + m\frac{V_0^2}{b}\delta$$

$$D = ma\ell$$



$$\frac{d\mathcal{M}_\xi}{dt} = T_\xi \quad \Rightarrow \quad J\frac{d^2\varphi}{dt^2} - \frac{ma\ell V_0}{b}\frac{d\delta}{dt} = mg\ell\varphi + \frac{m\ell V_0^2}{b}\delta$$

Compare with inverted pendulum!

The Inverted Pendulum Model $\delta \to \varphi$

Linearized tilt dynamics

$$J\frac{d^2\varphi}{dt^2} - \frac{ma\ell V_0}{b}\frac{d\delta}{dt} = mg\ell\varphi + \frac{m\ell V_0^2}{b}\delta$$

Model that relates steering angle δ to tilt φ

$$\frac{d^2\varphi}{dt^2} - \frac{mg\ell}{J}\varphi = \frac{m\ell V_0^2}{bJ}\delta + \frac{am\ell V_0}{bJ}\frac{d\delta}{dt}$$

Transfer function: $P(s) = \dfrac{am\ell V_0}{bJ}\,\dfrac{s + V_0/a}{s^2 - mg\ell/J}$

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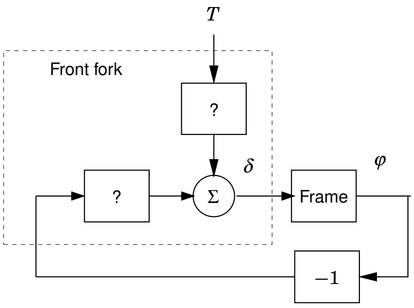
Some Interesting Questions

- How do you stabilize a bicycle?
  - By steering or by leaning?
- Do you normally stabilize a bicycle when you ride it?
- Why is it possible to ride no hands
- How is stabilization influenced by the design of the bike?
- Why does the front fork look the way it does?
- The main message:
  - A bicycle is a feedback system!
  - The front fork is the key!
- Is the control variable steering angle or steering torque?

Block Diagram of a Bicycle

Control variable: Handlebar torque T

Process variables: Steering angle δ, tilt angle φ

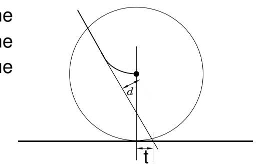


A feedback system

The Front Fork

The front fork has many interesting features that were developed over a long time. Its behavior is complicated by geometry, the trail, tire-road interaction and gyroscopic effects. We will describe it by a strongly simplified static linear model.

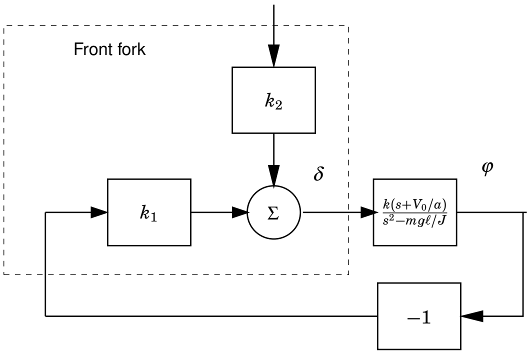
With a positive trail the front wheel lines up with the velocity (caster effect). The trail also creates a torque that turns the front fork into the lean. A static torque balance gives



$$T - mgt\varphi - mgt\alpha\delta = 0$$
$$\delta = -k_1\varphi + k_2 T$$

Qualitative experimental verification. In reality more complex, dynamics and velocity dependence will be discussed later.

Block Diagram of a Bicycle



The Closed Loop System

Combining the equations for the frame and the front fork gives

$$\frac{d^2\varphi}{dt^2} = \frac{mg\ell}{J}\varphi + \frac{am\ell V_0}{bJ}\frac{d\delta}{dt} + \frac{m\ell V_0^2}{bJ}\delta$$
$$\delta = -k_1\varphi + k_2 T$$

we find that the closed loop system is described by

$$\frac{d^2\varphi}{dt^2} + \frac{am\ell k_1 V_0}{bJ}\frac{d\varphi}{dt} + \frac{mg\ell}{J}\Big(\frac{k_1 V_0^2}{bg} - 1\Big)\varphi = \frac{amk_2\ell V_0}{bJ}\Big(\frac{dT}{dt} + \frac{V_0}{a}T\Big)$$

This equation is stable if

$$V_0 > V_c = \sqrt{bg/k_1}$$

where V_c is the critical velocity. Physical interpretation. Think about this next time you bike!

Stabilization

The bicycle is a feedback system. The clever design of the front fork gives a feedback because a the front wheel will steer into a lean. The closed loop system can be described by the equation

$$\frac{d^2\varphi}{dt^2}+\frac{am\ell k_1 V_0}{bJ}\frac{d\varphi}{dt}+\frac{mg\ell}{J}\Big(\frac{k_1 V_0^2}{bg}-1\Big)\varphi=\frac{amk_2\ell V_0}{bJ}\Big(\frac{dT}{dt}+\frac{V_0}{a}T\Big)$$

which shows how tilt angle φ depends on handle bar torque T.

The equation is unstable for low speed but stable for high speed $V_0 > V_c = \sqrt{bg/k_1}$, the critical velocity.

This means that the bicycle is self-stabilizing if the velocity is larger than the critical velocity V_c! You can observe this by rolling a bicycle down a gentle slope or by biking at different speeds.

Gyroscopic Effects

Gyroscopic effects has a little influence on the front fork

$$\frac{d^2\varphi}{dt^2}=\frac{mg\ell}{J}\varphi+\frac{am\ell V_0}{bJ}\frac{d\delta}{dt}+\frac{m\ell V_0^2}{bJ}\delta$$

$$\delta=-k_1\varphi-k_g\frac{d\varphi}{dt}+k_2T$$

we find that the closed loop system is described by

$$\Big(1+\frac{am\ell k_g V_0}{bJ}\Big)\frac{d^2\varphi}{dt^2}+\Big(\frac{am\ell k_1 V_0}{bJ}+\frac{m\ell k_g V_0^2}{bg}\Big)\frac{d\varphi}{dt}$$
$$+\frac{mg\ell}{J}\Big(\frac{k_1V_0^2}{bg}-1\Big)\varphi=\frac{amk_2\ell V_0}{bJ}\Big(\frac{dT}{dt}+\frac{V_0}{a}T\Big)$$

Damping is improved, but the stability condition is the same as before

$$V_0 > V_c = \sqrt{bg/k_1}$$

Lund Bicycle with Strong Gyroscopic Action

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Rear Wheel Steering

F. R. Whitt and D. G. Wilson (1974) Bicycling Science - Ergonomics and Mechanics. MIT Press Cambridge, MA.

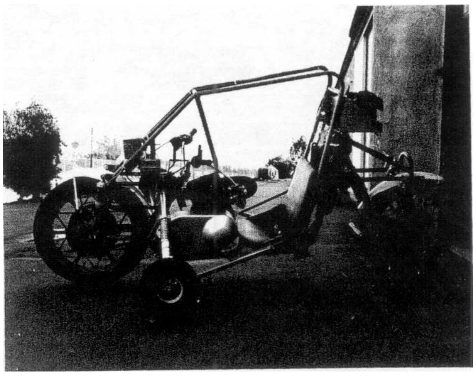
Many people have seen theoretical advantages in the fact that front-drive, rear-steered recumbent bicycles would have simpler transmissions than rear-driven recumbents and could have the center of mass nearer the front wheel than the rear. The U.S. Department of Transportation commissioned the construction of a safe motorcycle with this configuration. It turned out to be safe in an unexpected way: No one could ride it.

The Santa Barbara Connection

The NHSA Rear Steered Motorcycle

- The National Highway Safety Administration had a project aimed at developing a safe motorcycle in the late 1970s.
- Low center of mass
- Long wheel base
- Separation of steering and braking
- Robert Schwarz, South Coast Technology in Santa Barbara, California
- Use Sharp model reverse velocity
- Linearize analyse eigenvalues, in the range of 4 to 12 s for speeds ranging from 3 to 50 m/s
- Pointless to do experiments
- NHSA insisted on experiments

The NHSA Rear Steered Motorcycle



Comment by Robert Schwarz

The outriggers were essential; in fact, the only way to keep the machine upright for any measurable period of time was to start out down on one outrigger, apply a steer input to generate enough yaw velocity to pick up the outrigger and then attempt to catch it as the machine approached vertical. Analysis of film data indicated that the longest stretch on two wheels was about 2.5 s.

The Linearized Tilt Equation

Front wheel steering:

$$\frac{d^2\varphi}{dt^2} = \frac{mg\ell}{J}\varphi + \frac{am\ell V_0}{bJ}\frac{d\delta}{dt} + \frac{m\ell V_0^2}{bJ}\delta$$

Rear wheel steering (change sign of V_o):

$$\frac{d^2\varphi}{dt^2} = \frac{mg\ell}{J}\varphi - \frac{am\ell V_0}{bJ}\frac{d\delta}{dt} + \frac{m\ell V_0^2}{bJ}\delta$$

The transfer function of the system is

$$P(s) = \frac{am\ell V_0}{bJ}\frac{-s + \frac{V_0}{a}}{s^2 - \frac{mg\ell}{J}}$$

One pole and one zero in the right half plane.

The Transfer Function

$$P(s) = \frac{am\ell V_0}{bJ}\frac{-s + \frac{V_0}{a}}{s^2 - \frac{mg\ell}{J}}$$

One RHP pole at $p = \sqrt{\frac{mg\ell}{J}} \approx 3 \ \mathrm{rad/s}$ (the pendulum pole)

One RHP zero at $z = \frac{V_0}{a} \approx 5, \quad \frac{z}{p} = \frac{5}{3} \approx 1.7$, quad $M_s \geq 4$

Pole position independent of velocity but zero proportional to velocity. When velocity increases from zero to high velocity you pass a region where $z = p$ and the system is unreachable.

Does Feedback from Rear Fork Help?

Combining the equations for the frame and the rear fork gives

$$\frac{d^2\varphi}{dt^2} = \frac{mg\ell}{J}\varphi - \frac{am\ell V_0}{bJ}\frac{d\delta}{dt} + \frac{m\ell V_0^2}{bJ}\delta$$
$$\delta = -k_1\varphi + k_2 T$$

we find that the closed loop system is described by

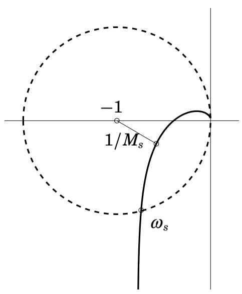
$$\frac{d^2\varphi}{dt^2} - \frac{am\ell k_1 V_0}{bJ}\frac{d\varphi}{dt} + \frac{mg\ell}{J}\left(\frac{k_1 V_0^2}{bg} - 1\right)\varphi = \frac{amk_2\ell V_0}{bJ}\left(\frac{dT}{dt} + \frac{V_0}{a}T\right)$$

where $V_c = \sqrt{bg/k_1}$. This equation is unstable for all k_1. There are several ways to turn the rear fork but it makes little difference.

Can the system be stabilized robustly with a more complex controller?

Can a general linear controller help?

Nyquist's stability theorem



The sensitivity function

$$S = \frac{1}{1+L}$$

For a system with a pole p and a zero z in the right half plane the maximum modulus theorem implies

$$M_s = \max_{\omega} |S(i\omega)| \geq \frac{|z+p|}{|z-p|}$$

$|S(i\omega)| < 2$ implies $z > 3p$ (or $z < p/3$) for any controller!

Return to Rear Wheel Steering ...

The zero-pole ratio is

$$\frac{z}{p} = \frac{V_0\sqrt{J}}{a\sqrt{mg\ell}} = \frac{V_0\sqrt{J_{cm} + m\ell^2}}{a\sqrt{mg\ell}}$$

The system is not controlable if $z = p$, and it cannot be controlled robustly if the ratio z/p is in the range of 0.3 to 3.

To make the ratio large you can

- Make a small by leaning forward
- Make V_0 large by biking fast (takes guts)
- Make J large by standing upright
- Sit down, lean back when the speed is sufficiently large

Klein's Un-ridable Bike

Klein's Ridable Bike

Experiments

Many interesting experiments can be performed with bicycles.

- Front fork model
  Ride in a straight line lean the body in one direction and determine the steer-torque required to maintain a straight line path.
- Stabilization
  - Push an riderless bicycle down a slope which gives the bicycle critical speed. Observe self-stabilization and investigate effects of trail and front-wheel inertia.
- Steering
  - Give a riderless bicycle a push on a flat surface. Apply a steering torque and observe the trajectory.

Instrumentation

Equipment can range from rudimentary to advanced

- Rudimentary
  A torque wrench to measure steer-torque, lean and speed sensors
- Intermediate
  Sensors for bike and rider lean, speed, steer torque with interfaces and a data logger
- Advanced
  A fully instrumented bicycle robot with electric drive motor and retractable support wheels. Cameras on bike and on the ground.

The Lund University Un-ridable Bike

The UCSB Rideable Bike

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Steering and Stabilization - A Classic Problem

Lecture by Wilbur Wright 1901:

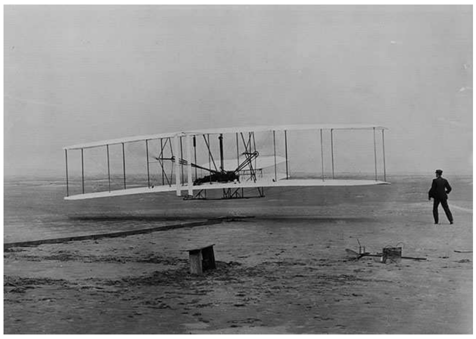
Men know how to construct air-planes.
Men also know how to build engines.
Inability to balance and steer still confronts
students of the flying problem.
When this one feature has been worked out,
the age of flying will have arrived, for
all other difficulties are of minor importance.

The Wright Brothers figured it out and flew the Wright Flyer at Kitty Hawk on December 17 1903!

Minorsky 1922:

It is an old adage that a stable ship is difficult to steer.

The Wright Flyer - Unstable but Maneuvrable



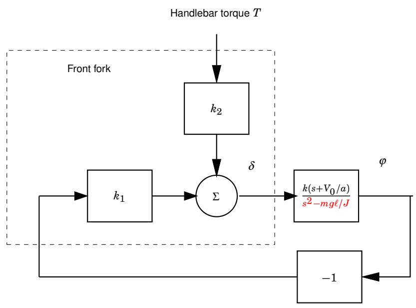
Steering

Having understood stabilization of bicycles we will now investigate steering for the bicycle with a rigid rider.

- Key question: How is the path of the bicycle influenced by the handle bar torque?
- Steps in analysis, find the relations
  - How handle bar torque influences steering angle
  - How steering angle influences velocity
  - How velocity influences the path

We will find that the instability of the bicycle frame causes some difficulties in steering (dynamics with right half plane zeros). This has caused severe accidents for motor bikes.

How Steer Torque Influences Steer Angle



Transfer function from T to δ is

$$\frac{k_2}{1+k_1P(s)} = \frac{k_2}{1+k_1\frac{k(s+V_0/a)}{s^2-mg\ell/J}} = k_2\frac{s^2-mg\ell/J}{s^2+\frac{am\ell k_1V_0}{bJ}s+\frac{mg\ell}{J}\left(\frac{V_0^2}{V_c^2}-1\right)}$$

Summary of Equations

Kinematics

$$\frac{dy}{dt} = V\psi$$
$$\frac{d\psi}{dt} = \frac{V}{b}\delta.$$

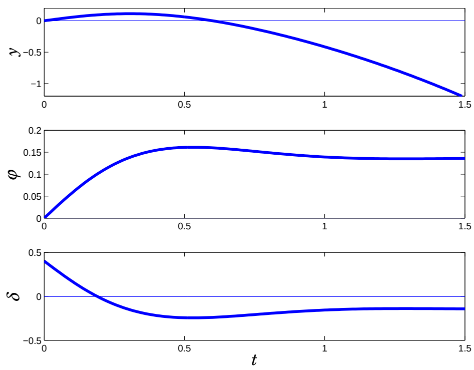
The transfer function from steer angle δ to path deviation y is

$$G_{y\delta}(s) = \frac{V^2}{bs^2}.$$

Transfer function from steer torque T to y

$$G_{yT}(s) = \frac{k_1 V^2}{b}\frac{s^2 - mgh/J}{s^2\left(s^2 + \frac{k_2 V D}{bJ}s + \frac{mgh}{J}(\frac{V^2}{V_c^2} - 1)\right)}.$$

Simulation



Summary

- The simple inverted pendulum model with a rigid rider can explain stabilization. The model indicates that steering is difficult due to the right half plane zero in the transfer function from handle bar torque to steering angle.
- The right half plane zero has some unexpected consequences which gives the bicycle a counterintuitive behavior. This has caused many motorcycle accidents.
- How can we reconcile the difficulties with our practical experience that a bicycle is easy to steer?
- The phenomena depends on the assumption that the rider does not lean.
- The difficulties can be avoided by introducing an extra control variable (leaning).

Coordinated Steering

An experienced rider uses both lean and the torque on the handle bar for steering. Intuitively it is done as follows:

- The bicycle is driven so fast so that it is automatically stabilized.
- The turn is initiated by a torque on the handle bar, the rider then leans gently into the turn to counteract the centripetal force which will tend to lean the bike in the wrong direction. This is particularly important for motor bikes which are much heavier than the rider.

A proper analysis of a bicycle where the rider leans require a more complex model because we have to account for two bodies instead of one. There are also two inputs to deal with. Accurate modeling of a bicycle also has to consider tire road interaction and a more detailed account of the mechanics.

Wilbur Wright on Counter-Steering

I have asked dozens of bicycle riders how they turn to the left. I have never found a single person who stated all the facts correctly when first asked. They almost invariably said that to turn to the left, they turned the handlebar to the left and as a result made a turn to the left. But on further questioning them, some would agree that they first turned the handlebar a little to the right, and then as the machine inclined to the left they turned the handlebar to the left and as a result made the circle inclining inwardly.

Wilbur's understanding of dynamics contributed significantly to the Wright brothers' success in making the first airplane flight.

Adding an input (lean) eliminates the RHP zero!

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Review

So far we have used a very simple second order model consisting of

- A momentum balance for frame and rider
- An empirical static model for the front fork

This led to the important observation that the front fork creates a feedback that can stabilize the system. It is natural to consider more complex models. A natural first step is to replace the static front fork model with a dynamic model. The closed loop system is then of fourth order, the linear version is Whipple's model.

Deriving the models is straight forward in principle but complexity rises quickly and calculations are error prone. Modeling software (multi-body software, Modelica) a great help.

Models of Increasing Complexity

- Second order linear model
- Fourth order linear model
- Fourth order nonlinear model
- Flexible tires
- Tire road interaction
- Frame flexibility
- Rider model
- Multi-body software useful
- There is a Modelica library for bicycles

Carvallo-Whipple 4th Order Linear Model

This model can be derived in different ways, Newton's equations, Lagrange's equations, projection methods etc. Calculations are complicated and error prone. Versions of the model are found in

- Whipple 1899
- Carvallo 1897-1900
- Klein and Sommerfeld 1910
- Neimark Fufaev 1968
- Many doctoral theses 1970-1990
- Schwab et al 2004

Parameters for 4th Order Linear Model

The model is described by 25 parameters; wheel base $b = 1.00$ m, trail $c = 0.08$, head angle $\lambda = 70°$, wheel radii $R_{rw} = R_{fw} = 0.35$, and data in the table.

	Rear Frame	*Fr Frame*	*Rr Wheel*	*Fr Wheel*
Mass m [kg]	87 (12)	2	1.5	1.5
Center of Mass				
x [m]	0.492 (0.439)	0.866	0	b
z [m]	1.028 (0.579)	0.676	R_{rw}	R_{fw}
Inertia Tensor				
J_{xx} [kg m^2]	3.28 (0.476)	0.08	0.07	0.07
J_{xz} [kg m^2]	-0.603 (-0.274)	0.02	0	0
J_{yy} [kg m^2]	3.880 (1.033)	0.07	0.14	0.14
J_{zz} [kg m^2]	0.566 (0.527)	0.02	J_{xx}	J_{xx}

A Fourth Order Linear Model

Momentum balances for frame and front fork

$$M\begin{pmatrix}\ddot{\varphi}\\ \ddot{\delta}\end{pmatrix} + CV\begin{pmatrix}\dot{\varphi}\\ \dot{\delta}\end{pmatrix} + (K_0 + K_2V^2)\begin{pmatrix}\varphi\\ \delta\end{pmatrix} = \begin{pmatrix}0\\ T\end{pmatrix},$$

Notice structure of velocity dependence. The matrices are

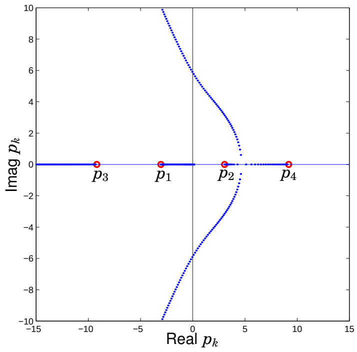
$$M = \begin{pmatrix}96.8\ (6.00) & -3.57(-0.472)\\ -3.57\ (-0.472) & 0.258\ (0.152)\end{pmatrix},$$

$$C = \begin{pmatrix}0 & -50.8\ (-5.84)\\ 0.436\ (0.436) & 2.20\ (0.666)\end{pmatrix},$$

$$K_0 = \begin{pmatrix}-901.0\ (-91.72) & 35.17\ (7.51)\\ 35.17\ (7.51) & -12.03\ (-2.57)\end{pmatrix},$$

$$K_2 = \begin{pmatrix}0 & -87.06\ (-9.54)\\ 0 & 3.50\ (0.848)\end{pmatrix}.$$

Root Locus Bicycle with Rider



Movies of Weave and Wobble

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Robot Bicycles

- 1988 Klein UIUC
- 1996 Pacejka Delft mmotorcycle robot
- 2004 Tanaka and Murakami
- 2005 UCSB
- 2005 Yamakita and Utano Titech
- 2005 Murata Co

Murata Manufacturing Company
Japan Times Oct 5 2005

Klein's Adapted Bikes for Children with Disabilities

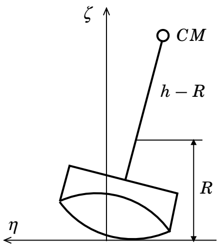
Over a dozen clinics for children and adults with a wide range of disabilities, including Down syndrome, autism, mild cerebral palsy and Asperger's syndrome. More than 2000 children aged 6-20 have been treated, see

http://www.losethetrainingwheels.org

Scaled Dynamics

Unstable pole

$$p = \sqrt{\frac{mg(h-R)}{J}} \approx \sqrt{\frac{(1-R/h)g}{h}}.$$



Critical velocity

$$V_c = \sqrt{\frac{gb(h-R)}{k_2 h}}$$

Same behavior as an ordinary bike but dynamics is slower and more stable. The children learn the right behavior in a gentle environment, the dynamics is then gradually speeded up to that of a normal bike.

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Conclusions

- Bicycle dynamics is a good illustration theoretically and experimentally
  - Much insight into stabilization and steering can be derived from simple models
  - Interaction of system and control design (the front fork)
  - Counterintuitive behavior because of dynamics with right half plane zeros
  - Importance of several control variables
- Lesson 1: Dynamics is important! Things may look OK statically but intractable because of dynamics.
- Lesson 2: A system that is difficult to control because of zeros in the right half plane can be improved significantly by introducing more control variables.

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